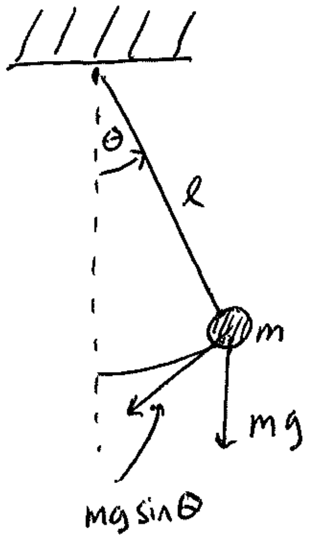
1.3 The Pendulum → measurement of time!

Simple pendulum: Point mass m on massless rigid rod



- Displacement along arc: $\ell\theta$
- Angular velocity: $\dfrac{d\theta}{dt}$
- Angular acceleration: $\dfrac{d^2\theta}{dt^2}$
- Newton's law: $F = ma = m\ell\dfrac{d^2\theta}{dt^2} = -mg\sin\theta$

minus b/c restoring force back to EP

$$\rightarrow \frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\sin\theta$$

- This eq. does not have form of SHO!
- But for small θ, $\sin\theta \simeq \theta - \dfrac{\theta^3}{3!} + \dfrac{\theta^5}{5!} + \ldots$

$$\rightarrow \boxed{\frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\theta}$$

SHO! $\omega = \sqrt{\dfrac{g}{\ell}}$

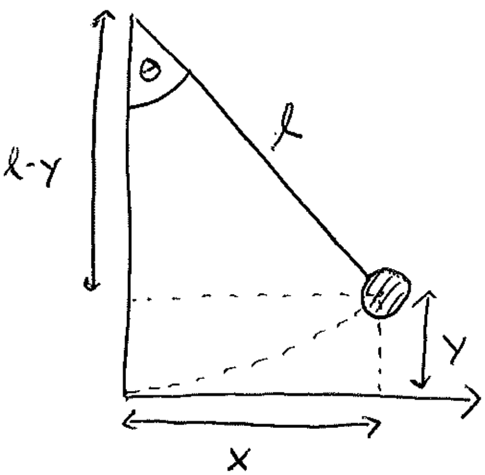
$T = 2\pi\sqrt{\ell/g}$ → independent of mass & amplitude!

- General solution: $\theta(t) = \theta_0 \cos(\omega t + \phi)$

(θ_0:) angular amplitude

- Measuring T yields g if ℓ is known. With $g = 9.81\ m/s^2$ and $\ell = 1\ m$, period $T = 2.006$ seconds

Energy of a simple pendulum



- $x = l\sin\theta \sim l\theta$ for small θ

- $y = l - l\cos\theta$

  with $\cos\theta \simeq 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots$

  $= \frac{l\theta^2}{2}$

- use $x = l\sin\theta \sim l\theta \rightarrow \theta = \frac{x}{l} \Rightarrow y \simeq \frac{x^2}{2l}$

- Potential energy $U = mgy = mg\frac{x^2}{2l}$

- Total energy $E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}mg\frac{x^2}{l} = \frac{1}{2}mg\frac{A^2}{l}$ for all times

  where A is max amplitude, $x = A$, $v = 0$

- From energy, can derive velocity and displacement:

$$v(x) = \frac{dx}{dt} = \sqrt{\frac{g(A^2 - x^2)}{l}}$$ (using total energy above)

$$\rightarrow \int \frac{dx}{\sqrt{A^2 - x^2}} = \sqrt{\frac{g}{l}} \int dt$$

use $x = A\sin\theta$

$$\arcsin\left(\frac{x}{A}\right) = \sqrt{\frac{g}{l}}\,t + \underbrace{\phi}_{\text{constant of integration}}$$

$$\rightarrow x = A\sin\left(\sqrt{\frac{g}{l}}\,t + \phi\right)$$ → same as before!

We derived trajectory purely from energetic considerations!

Comparison:

Mass on spring: $E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$

Simple pendulum: $E = \frac{1}{2}mv^2 + \frac{1}{2}\frac{mg}{\ell}x^2$

General form: $E = \frac{1}{2}\alpha v^2 + \frac{1}{2}\beta x^2$

related to inertia

restoring force / distance x

- For all SHOs the equations have the same form, only the labels of physical quantities change
- No need to repeat analysis for each system, results can be transferred

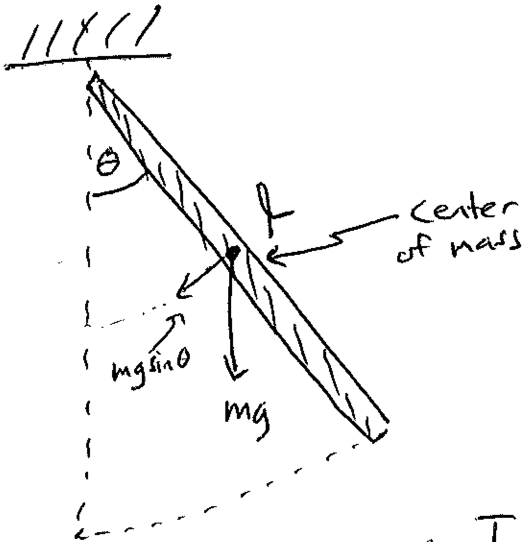
$$\rightarrow \frac{dE}{dt} = \alpha v \frac{dv}{dt} + \beta x \frac{dx}{dt} = 0$$

$$\rightarrow \boxed{\frac{d^2x}{dt^2} = -\frac{\beta}{\alpha}x}$$

identical to earlier form w/ $\omega = \sqrt{\beta/\alpha}$

Physical pendulum

- A real pendulum has ~~s~~ a rod w/ mass



- For rotating systems, Newton's law

$$F = m\frac{d^2x}{dt^2} \text{ becomes}$$

$$\boxed{I\frac{d^2\theta}{dt^2} = \tau}$$

I: moment of inertia

τ: applied torque

- $I = \frac{1}{3}m\ell^2$ for uniform rod

- Torque on rod when displaced through angle θ is given by torque arm length $\frac{1}{2}\ell$ multiplied by component of force normal to torque arm $-mg\sin\theta$

$$\rightarrow \tau = \left(\frac{\ell}{2}\right) \times (-mg\sin\theta)$$

$$\Rightarrow \frac{1}{3}m\ell^2\frac{d^2\theta}{dt^2} = -\frac{1}{2}mg\ell\sin\theta$$

$$\rightarrow \frac{d^2\theta}{dt^2} = -\frac{3g}{2\ell}\sin\theta \simeq -\frac{3}{2}\frac{g}{\ell}\theta \qquad \text{(small } \theta\text{)}$$

$$\rightarrow \omega = \sqrt{3g/2\ell} \qquad T = 2\pi\sqrt{2\ell/3g}$$

Example: Human leg! Comfortable walking at natural period

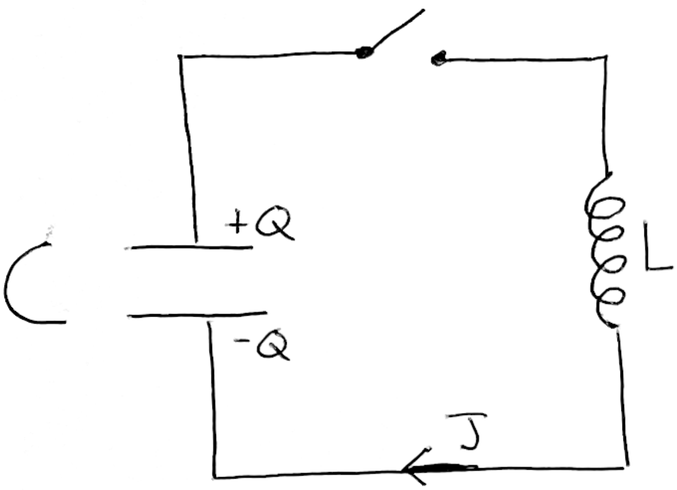
For $\ell = 0.8$ m, $T = 1.5$ s ($\simeq 2$ strides) $\Delta s = 1$ m/stride

$V_{walk} = 2\text{m}/1.5\text{s} \sim 4.8$ km/h ~ 3 mph!

1.4 Oscillations in electrical circuits

The LC Circuit

An LC circuit is the simplest example of an oscillating electrical circuit:



Capacitor C

Inductor L

Assume Resistance $R = 0$

→ Idealized situation w/ $R=0$, similar to Frictionless mechanical oscillator

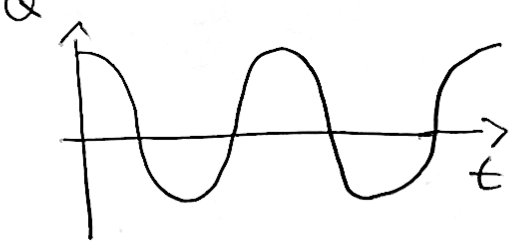
- Initially, capacitor charged with voltage V_c → $Q = V_c C$
- Next, switch is closed. Charge flows away from the charged capacitor to reduce its potential energy (voltage), ~~the inductor~~ causing a current $J \equiv \frac{dQ}{dt}$ to flow in the circuit
- The time-varying current $J(t)$ induces a voltage across the inductor: $V_L = L \frac{dJ}{dt}$
- We use Kirchhoff's Law (Sum of voltages around circuit vanishes):

$$0 = V_c + V_L = \frac{Q}{C} + L\frac{dJ}{dt} = \frac{Q}{C} + L\frac{d^2Q}{dt^2}$$

$$\rightarrow \boxed{\frac{d^2Q}{dt^2} = -\frac{1}{LC}Q}$$

Same form as differential eq. for mechanical oscillators!

· Solution: $Q(t) = \underbrace{Q_0}_{\text{initial charge on capacitor}} \cos \omega t$ with $\omega = \sqrt{\frac{1}{LC}}$



· The LC oscillator contains electrostatic energy (in capacitor) and magnetic energy (in inductor):

$$E = \frac{1}{2} L J^2 + \frac{1}{2} C V_c^2 = \frac{1}{2} L J^2 + \frac{1}{2} \frac{Q^2}{C}$$

· During oscillation there is a continuous exchange b/t electrostatic and magnetic energy.

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Similarities in Physics

· The mathematical description of mech. and electr. oscillators has identical form:

mech	electr.	general
$m \frac{d^2x}{dt^2} = -kx$	$L \frac{d^2Q}{dt^2} = -\frac{1}{C} Q$	$\alpha \frac{d^2 Z}{dt^2} = -\beta Z$
$E = \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 + \frac{1}{2} k x^2$	$E = \frac{1}{2} L \left(\frac{dQ}{dt}\right)^2 + \frac{1}{2} \frac{Q^2}{C}$	$E = \frac{1}{2} \alpha \left(\frac{dZ}{dt}\right)^2 + \frac{1}{2} \beta Z^2$

· Correspondence: $x \leftrightarrow Q$
$m \leftrightarrow L$
$k \leftrightarrow 1/C$

$Z = Z(t)$ is oscillating quantity.

α, β are constants

· An electr. oscillator can simulate the physics of a mech. oscillator and can serve as an "analog computer"

· Generally: If mathematical description of two physical systems is the same, then one can be used to simulate the other

→ True for classical and quantum systems!